# Quantum Field Theory

## Set 12

### Exercise 1: Compton Scattering in the rest frame of the initial electron

The aim of this exercise is to compute the differential cross section $\frac{d\sigma}{d\cos\theta}$ for the process $e^-\gamma \longrightarrow e^-\gamma$.
We will proceed through the following steps:

- Preliminary: deduce the famous Compton relation between the energies of the initial and final photon in the rest frame of the incoming electron.
- Draw all the Feynman diagrams contributing to the scattering.
- Using the Feynman rules for QED, write the expression for the amplitude $i\mathcal{M} \equiv i\mathcal{M}^{\mu\nu}\epsilon_\nu\epsilon^*_\mu$, where $\epsilon_\nu$ and $\epsilon^*_\mu$ are the polarizations of the incoming and outgoing photon.
- Verify the gauge invariance of the matrix element by checking explicitly that $k_\nu\mathcal{M}^{\mu\nu} = k'_\mu\mathcal{M}^{\mu\nu} = 0$, where $k^\mu$ $(k'^\mu)$ is the momentum of the incoming (outgoing) photon.
- Obtain the unpolarized squared amplitude by summing over all polarizations of the final particles and averaging over all polarizations of the initial particles, by making use of the formulas

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m, \qquad\qquad \sum_{pol}\epsilon_\mu\epsilon^*_\rho \to -\eta_{\mu\rho},$$

  where the symbol '$\to$' means that the replacement holds up to longitudinal terms (vanishing in the amplitude by gauge invariance).
- Simplify the obtained expression using the identities:

$$\begin{aligned} \gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu &= -2\not{c}\not{b}\not{a}, \\ \gamma^\mu \not{a}\not{b}\gamma_\mu &= 4\, a\cdot b, \\ \gamma^\mu \not{a}\gamma_\mu &= -2\not{a}, \\ \gamma^\mu\gamma_\mu &= 4, \end{aligned}$$

  and reduce the traces to a maximum of 4 Dirac matrices, which you can evaluate using

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] = 4\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right).$$

- Show that the final result is

$$\frac{1}{4}\sum_{pol}|\mathcal{M}^{\mu\nu}\epsilon_\nu\epsilon^*_\mu|^2 = 2e^4\left[\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) + m^4\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right)^2\right],$$

  where we have denoted with $p^\mu$ $(p'^\mu)$ the 4-momentum of the incoming (outgoing) electron and with $k\,(k')$ the 4-momentum of the incoming (outgoing) photon.
- Making use of the expression for the 2-body phase space compute the differential cross section $\frac{d\sigma}{d\cos\theta}$.
- Finally study the high energy ($s \gg m^2$) and low energy ($s \ll m^2$) limits, where $s \equiv (p+k)^2$. Show that in the latter case the total cross section reduces to the Thompson cross section $\sigma_T = \frac{8\pi}{3}\frac{\alpha^2}{m^2}$ where $\alpha = e^2/(4\pi)$.